Problem set 1
Due date: 21st Jan

Part A (submit any four)

Exercise 1.
(1) $\mathcal{F}$ is closed under finite and countable unions, intersections, differences, symmetric differences.
(2) If $A_n \in \mathcal{F}$, $n = 1,2\dots$, then $\limsup A_n := \{\omega \,:\, \omega$ belongs to infinitely many $A_n\}$ and $\liminf A_n := \{\omega \,:\, \omega$ belongs to all but finitely many $A_n\}$ are also in $\mathcal{F}$. In particular, if A_n increases or decreases to A, then $A \in \mathcal{F}$.
(3) $\mathbf{P}(\phi) = 0$, $\mathbf{P}(\Omega) = 1$. For any $A, B \in \mathcal{F}$ we have $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. If $A_n \in \mathcal{F}$, then $\mathbf{P}(\cup A_n) \leq \sum \mathbf{P}(A_n)$.
(4) If $A_n \in \mathcal{F}$ and A_n increases (decreases) to A, the $\mathbf{P}(A_n)$ increases (decreases) to $\mathbf{P}(A)$.

Exercise 2. Let Ω be an arbitrary set. Let $\mathcal{F} = \{A \subset \Omega \,:\,$ either A or A^c is countable$\}$ (where 'countable' includes finite and empty sets). Define $\mathbf{P}(A) = 0$ if A is countable and $\mathbf{P}(A) = 1$ if A^c is countable. Show that $\mathcal{F}$ is a σ-field and that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

Exercise 3. Suppose S is a π-system and is further closed under complements ($A \in S$ implies $A^c \in S$). Show that S is an algebra.

Exercise 4. Let $\mathbf{P}$ be a p.m. on a σ-algebra $\mathcal{F}$ and suppose $S \subset \mathcal{F}$ be a π-system. If $A_k \in S$ for $k \leq n$, write $\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n)$ in terms of probabilities of sets in S.

Exercise 5. Suppose $\sigma(S) = \mathcal{F}$ and $\mathbf{P}, \mathbf{Q}$ are two probability measure on $\mathcal{F}$. If $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in S$, is it necessarily true that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$? If yes, prove it. If not, give a counterexample.

Exercise 6. Let m_* be the Lebesgue outer measure on $[0,1]$. Show that $m_*[a,b] = b - a$ for any $[a,b] \subset [0,1]$.

Part B (submit any one)

Exercise 7 (How much bigger is $\overline{\mathcal{B}}$ than $\mathcal{B}$?). Let m be the Lebesgue p.m. on the Cartheodary σ-algebra $\overline{\mathcal{B}}$ and let m_* be the corresponding outer measure defined on all subsets. We say that a subset $N \subset [0,1]$ is a null set if $m_*(N) = 0$. Show that

$$\overline{\mathcal{B}} = \{B \cup N \,:\, B \in \mathcal{B} \text{ and } N \text{ is null}\}$$

where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$.

Exercise 8 (Completion of σ-algebras). Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, say that N is $\mathbf{P}$-null if there is some $A \in \mathcal{F}$ with $A \supset N$ and $\mathbf{P}(A) = 0$. Define

$$\overline{\mathcal{F}} = \{B \cup N \,:\, B \in \mathcal{F} \text{ and } N \text{ is null}\}$$

and for any $A \in \overline{\mathcal{F}}$ set $\overline{\mathbf{P}}(A) = \mathbf{P}(B)$ if $A = B \cup N$ with $B \in \mathcal{B}$ and N is $\mathbf{P}$-null (first check that $\overline{\mathbf{P}}(A)$ is well-defined since there may be many ways to write A in the form $B \cup N$).

Show that $\overline{\mathcal{F}}$ is a σ-algebra and that $\overline{\mathbf{P}}$ a p.m. on $\overline{\mathcal{F}}$. [Remark: From this exercise and the last, $\overline{\mathcal{B}}$ is just the completion of $\mathcal{B}$]

Exercise 9 (Further non-trivial measures). Use the general procedure as described in the lectures to construct the following measures.

(a) A p.m. on $([0,1]^d, \mathcal{B})$ such that $\mathbf{P}([a_1,b_1] \times \ldots \times [a_d,b_d]) = \prod_{k=1}^{d}(b_k - a_k)$ for all cubes contained in $[0,1]^d$. This is the d-dimensional Lebesgue measure.

(b) A p.m. on $\{0,1\}^{\mathbb{N}}$ such that for any cylinder set $A = \{\omega \,:\, \omega_{k_j} = \varepsilon_j,\ j = 1,\ldots,n\}$ (any $n \geq 1$ and $k_j \in \mathbb{N}$ and $\varepsilon_j \in \{0,1\}$) we have (for a fixed $p \in [0,1]$ and $q = 1-p$)

$$\mathbf{P}(A) = \prod_{j=1}^{n} p^{\varepsilon_j} q^{1-\varepsilon_j}.$$

[**Hint:** Start with the algebra generated by cylinder sets].